Anyon Algebra

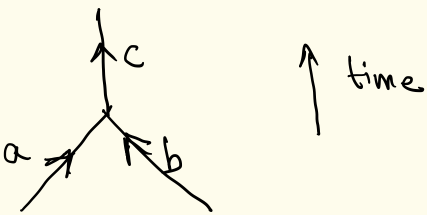
There are two basic operations in a system with anyons: fusion and braiding.

Fusion

Denoting anyon types by a, b, c ..., we write

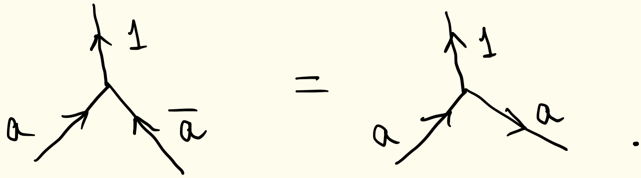
$$a \times b = \sum_c N^c_{ab} \, c$$

, which means that a and b fuse to give a linear combination of anyon types. N^c_{ab} are allowed to be greater than one, e.g., if anyons correspond to representations of SU(3), but we will restrict our focus to cases where $N^c_{ab} = 0$ or 1. One can depict a specific fusion channel as:

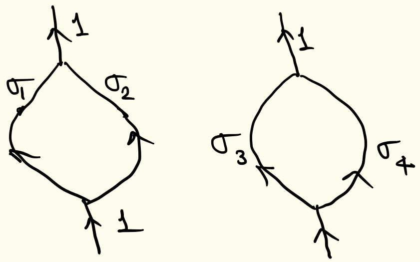


Each anyon a has its anti-particle $\bar{a}$, which could be the anyon itself. $a \times \bar{a} = 1 + \dots$

The anti-anyon propagating forward is identical to to the anyon moving backward in time

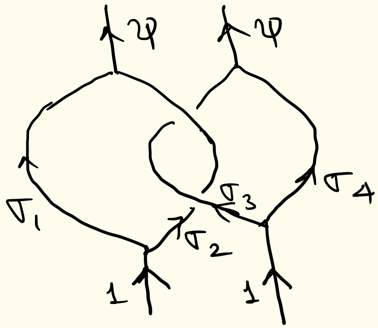


Locality and fusion: If a and b fuse to c, braiding a and b in any which way will not change the fusion channel. However, if a fourth anyon d first braids with a or b, and then a and b are fused, the fusion channel can change. For example, consider four Ising anyons pulled out of vacuum.



They fuse to one since fusion is a local property as just discussed.

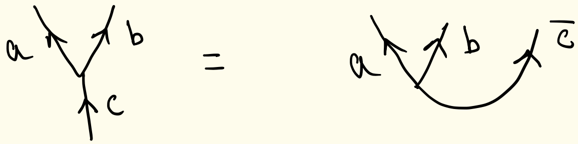
However if σ_2 and σ_3 are first braided then the outcome of fusion could be a ψ particle.



We will do an explicit calculation of this kind later.

Properties of fusion matrix N:

(a)



It's useful to define $N_{ab\bar{c}} = N^{c}_{ab}$ since $N_{\alpha\beta\gamma}$ is symmetric in all its indices.

Also $a \times b = N^{c}_{ab}\, c \Rightarrow \bar{a} \times \bar{b} = N^{c}_{ab}\, \bar{c}$

Such considerations yield $N^{c}_{ab} = N^{\bar{c}}_{\bar{a}\bar{b}} = N^{\bar{b}}_{a\bar{c}} = N^{c}_{\bar{a}b}$

(b) Associativity: $(a \times b) \times c = a \times (b \times c)$

$$\Rightarrow \sum_d N^{d}_{ab} N^{e}_{cd} = \sum_f N^{f}_{cb} N^{e}_{af}$$

$$\Rightarrow \sum_d [N_a]_{bd} [N_c]_{de} = \sum_f [N_c]_{bf} [N_a]_{fe}$$

where $[N_a]_{ij} \stackrel{\text{def.}}{=} N^{j}_{a\,i}$

$\Rightarrow \quad N_a N_c = N_c N_a$

$\Rightarrow$ All fusion matrices commute. It turns out that the matrix that diagonalizes $\{N\}$ matrices is the modular S-matrix whose elements S_{ij} are proportional to the mutual statistics (braiding) of anyon i with anyon j.

Change of basis: F symbols

Fusion is quite analogous to angular momentum addition. e.g., for Ising anyons $\sigma \times \sigma = 1 + \psi \equiv \frac{1}{2} \times \frac{1}{2} = 0 + \frac{1}{2}$. Consider adding three spin-$\frac{1}{2}$s : $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$. One can first add the first and second spin or alternatively, first add the second and third spin. The basis change between these two set of states is implemented via the Wigner 6j symbol. Same is true for fusion.

Example:
Consider four Ising anyons (Majorana zero modes), $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Fix the fermion parity $i\gamma_1\gamma_2 i\gamma_3\gamma_4 = 1$. Let's first consider the basis in which $i\gamma_1\gamma_2$ and $i\gamma_3\gamma_4$ are diagonal.

One can represent $i\gamma_1\gamma_2$ as pauli Z_1 and $i\gamma_3\gamma_4$ as pauli Z_2. Fermion parity constraint implies that $Z_1 Z_2 = 1$ $\Rightarrow$ the two dimensional space on which the Ising anyons operate is:

$$|\uparrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle.$$

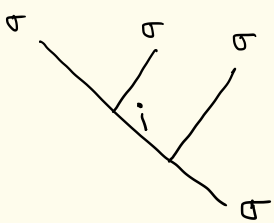
Now let's instead consider the basis in which $i\gamma_1\gamma_4$ and $i\gamma_2\gamma_3$ are diagonal. One can check that the following operator assignment works:

$$i\gamma_4\gamma_1 = X_1 X_2 \qquad i\gamma_2\gamma_3 = Y_1 Y_2$$

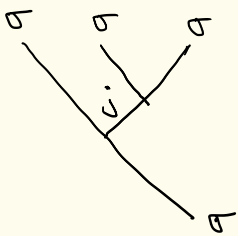
The eigenstates in this new basis are: $\frac{1}{\sqrt{2}}[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle]$, $\frac{1}{\sqrt{2}}[|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle]$. Thus, the matrix that implements the basis change is:

$$F = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

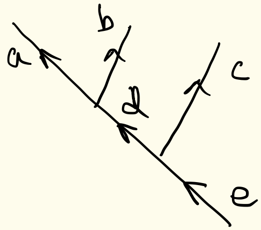
Pictorially:



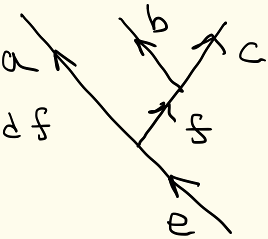
$$= \sum_j F_{ij}$$



More generally, the F-matrix is defined as:



$$= \sum_f \left[F^{abc}_{e}\right]^{a}_{df}$$



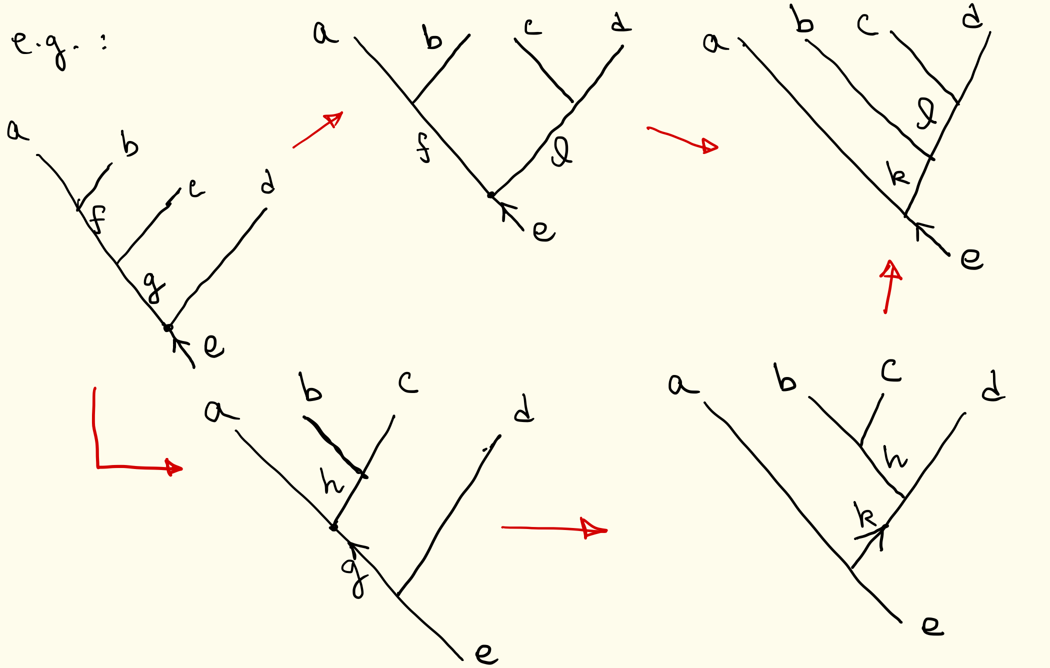
Thus, in the above example, we evaluated $F^{\sigma\sigma\sigma}_{\sigma}$.

Since the F matrix implements a basis change, it must be unitary when considered a 2×2 matrix with indices d, f above.

Pentagon Equation

The F-matrix allows one to transform one state into another in several different ways.

e.g.:



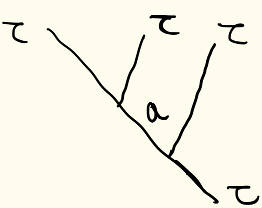
where we have suppressed several arrows for anyon trajectories to avoid clutter. Symbolically, this corresponds to

$$\left[F^{fcd}_{e}\right]_{gl} \left[F^{abl}_{e}\right]_{fk} = \sum_{h} \left[F^{abc}_{g}\right]_{fh} \left[F^{ahd}_{e}\right]_{gk} \left[F^{bcd}_{k}\right]_{hl}$$

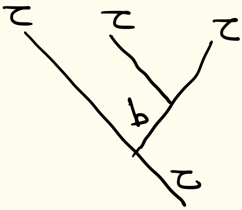
This equation puts a very severe constraint on the allowed values of F. For a given set of fusion rules, there are only a finite set of solutions. Combined with the hexagon eqn. (see below), one obtains essentially all known topological phases.

Application:

Lets use the pentagon eqn. to solve for the F-matrix for Fibonacci anyons, whose fusion rule is: $\tau \times \tau = 1 + \tau$. The interesting case is $F^{\tau\tau\tau}_{\tau}$, since then there is a two-dimensional subspace on which F acts.

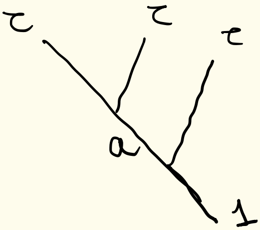


$$= \sum_b F_{ab}$$

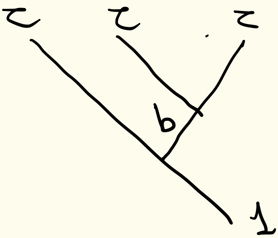


Both a, b are allowed to be 1 or τ.

In contrast consider $F^{\tau\tau\tau}_{1}$:



$$=$$



Here the only non-zero element corresponds to $a = \tau$, $b = \tau$, since $\tau \times 1 = \tau$.

One can similarly check other cases and conclude that the interesting case is $F^{\tau\tau\tau}_{\tau}$. To determine this, first set all indices except h in the pentagon eqn to τ $\Rightarrow$

$$\left[F^{\tau\tau\tau}_{\tau}\right]_{\tau\tau}\left[F^{\tau\tau\tau}_{\tau}\right]_{\tau\tau} = \sum_h \left[F^{\tau\tau\tau}_{\tau}\right]_{\tau h}\left[F^{\tau h \tau}_{\tau}\right]_{\tau\tau}\left[F^{\tau\tau\tau}_{\tau}\right]_{h\tau}$$

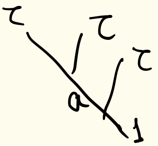
$$= \left[F^{\tau\tau\tau}_{\tau}\right]_{\tau\tau}\left[F^{\tau\tau\tau}_{\tau}\right]\left[F^{\tau\tau\tau}_{\tau}\right]_{\tau\tau}$$

$$+ \left[F^{\tau\tau\tau}_{\tau}\right]_{\tau 1}\left[F^{\tau 1 \tau}_{\tau\tau}\right]_{\tau\tau}\left[F^{\tau\tau\tau}_{\tau}\right]_{1\tau}$$

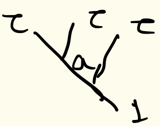
As discussed above $\left[F^{\tau 1 \tau}_{\tau\tau}\right]_{\tau\tau} = 1$.

$$\Rightarrow \left[F^{\tau\tau\tau}_{\tau}\right]^2_{\tau\tau} = \left[F^{\tau\tau\tau}_{\tau}\right]^3_{\tau\tau} + \left[F^{\tau\tau\tau}_{\tau}\right]_{1\tau}\left[F^{\tau\tau\tau}_{\tau}\right]_{\tau 1}$$

Next, consider, e.g. (one can try other combinations of course), $a=b=c=d=\tau$ and $e=1$. Since $F^{\tau\tau\tau}_{1}$ corresponds to



and is non-zero only when $a=\tau$ $\Rightarrow$ $g = l = \tau$. Now $F^{abl}_{e} = F^{\tau\tau\tau}_{1}$ corresponds to



and is non-zero only when $a=\tau$ $\Rightarrow$ $f = k = \tau$.

$$\Rightarrow \quad 1 = \sum_h \left[F^{\tau\tau\tau}_{\tau}\right]_{\tau h} \left[F^{\tau h \tau}_{1}\right]_{\tau\tau} \left[F^{\tau\tau\tau}_{\tau}\right]_{h\tau}$$

$$= \left[F^{\tau\tau\tau}_{\tau}\right]_{\tau 1} \left[F^{\tau\tau\tau}_{\tau}\right]_{1\tau} + \left[F^{\tau\tau\tau}_{\tau}\right]^2_{\tau\tau}$$

Taking $F^{\tau\tau\tau}_{\tau}$ to be real (this is a guage choice),

Unitarity implies
$$\underbrace{\begin{bmatrix} F_{11} & F_{1\tau} \\ F_{\tau 1} & F_{\tau\tau} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} F_{11} & F_{\tau 1} \\ F_{1\tau} & F_{\tau\tau} \end{bmatrix}}_{U^{\dagger}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad F_{11}^2 + F_{1\tau}^2 = 1 \qquad F_{11} F_{\tau 1} + F_{1\tau} F_{\tau\tau} = 0$$

$$F_{\tau\tau}^2 + F_{\tau 1}^2 = 1$$

Solving these, $F_{11} = -F_{\tau\tau}$, $F_{1\tau} = F_{\tau 1}$.

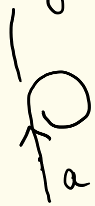
The eqn for $F_{\tau\tau}$ is : $F^3_{\tau\tau} = 1 - 2F^2_{\tau\tau}$.

The only soln. compatible with unitarity is the (golden mean)$^{-1}$, $F_{\tau\tau} = \frac{\sqrt{5}-1}{2} = \varphi^{-1}$. Solving the rest

$$\boxed{F^{\tau\tau\tau}_{\tau} = \begin{bmatrix} \varphi^{-1} & \sqrt{\varphi^{-1}} \\ \sqrt{\varphi^{-1}} & -\varphi^{-1} \end{bmatrix}}$$

Braiding and Self-Statistics

The self-statistics of anyons is obtained by twisting the world-line by 2π:

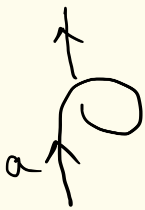


$= e^{2\pi i h_a}$

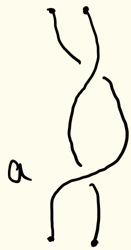
$h_a = 0$ for bosons,

$h_a = 1/2$ for fermions.

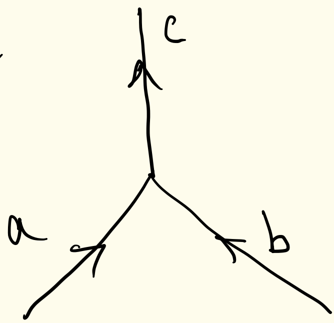
Thinking of worldline as a ribbon, this is same as twisting the ribbon:



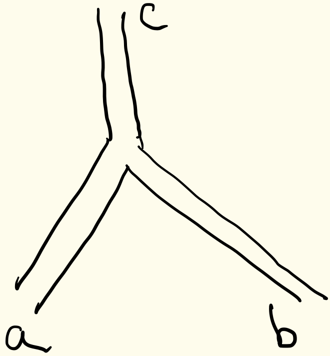
$\equiv$



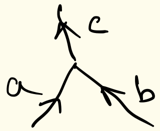
Now, let's consider two anyons fusing into a third one

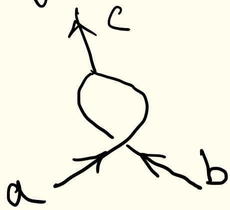


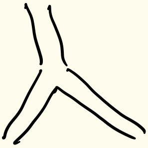
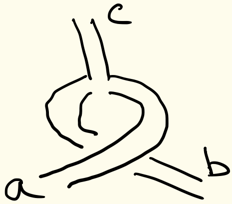
$\equiv$



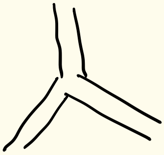
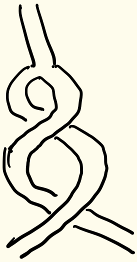
Lets define the braiding operation R^c_{ab} as

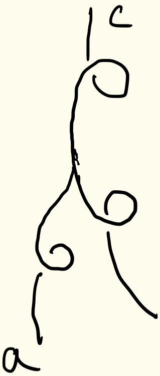
R^c_{ab} 

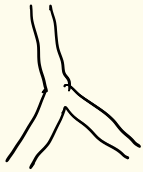
$=$ 

equivalently, R^c_{ab}  $=$ 

One naively may think that braiding twice is related to self-twist. Lets see how:

$R^c_{ab}\, R^c_{ba}$  $=$ 

$=$ 

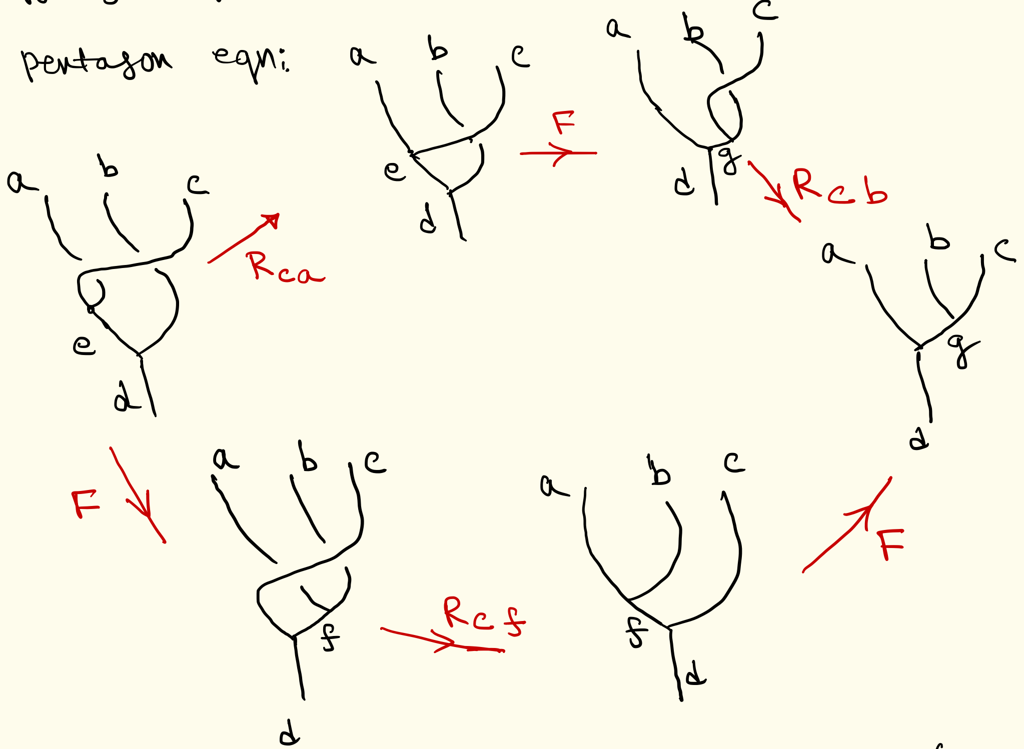
$= e^{2\pi i (h_c - h_a - h_b)}$ 

$$\Rightarrow R^c_{ab}\, R^c_{ba} = e^{2\pi i (h_c - h_a - h_b)}.$$

(try this using a paper cut like the ribbon above).

Hexagon Eqn.

A combination of R and F operations leads to a constraint in similar spirit as the pentagon eqn:



We will skip the explicit form with all indices (see Steve Simon's book). Schematically,

$$R^{ca}_{e} \; F \; R^{cb}_{g} = \sum_{f} F \; R^{cf}_{d} \; F$$

There is a similar eqn with R^{-1} instead of R.

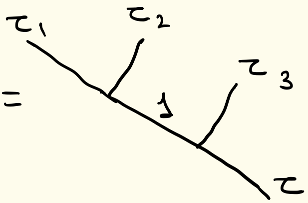
$$R^{-1} F R^{-1} = \sum F R^{-1} F.$$

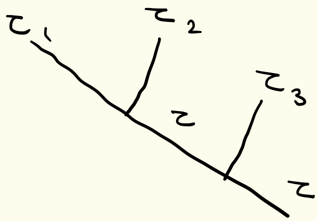
Solving pentagon and hexagon eqns. is difficult. It is generally believed that solving these eqns. yields all anyonic theories in 2+1-d.

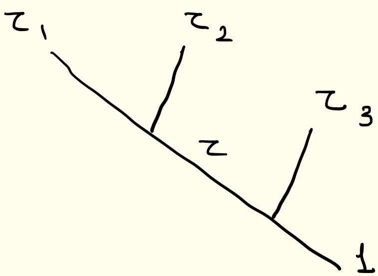
An application: Braid matrices for Fib. anyons.

R and F matrices are especially helpful to express braiding of anyons which do not have a fusion channel in terms of simpler processes.

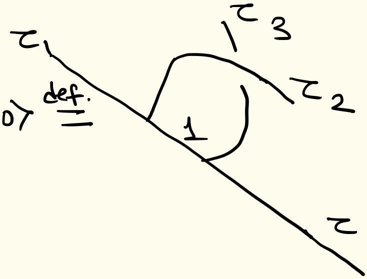
Consider three Fib. anyons: The Hilbert space is spanned by following three vectors:

$|0\rangle =$ 

$|1\rangle =$ 

$|2\rangle =$ 

Lets consider a process where τ_2 and τ_3 are braided in state $|0\rangle$:

$\hat{B}_{23}|0\rangle \overset{\text{def.}}{=}$ 

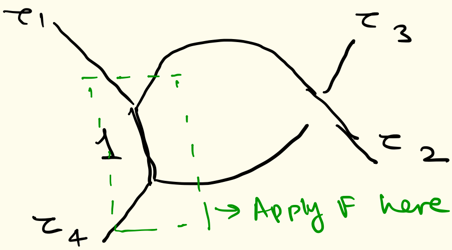
This defines the braid matrix

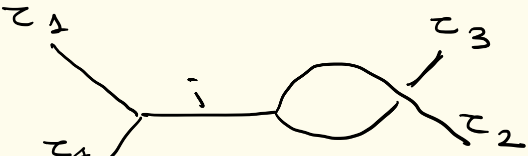
$$\hat{B}_{23}|0\rangle = (B_{23})_{11}|0\rangle + (B_{23})_{1\tau}|1\rangle$$

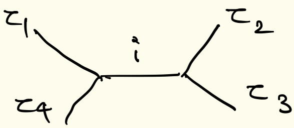
Similarly,

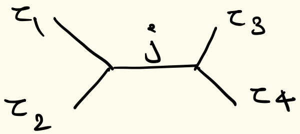
$$\hat{B}_{23}|1\rangle = (B_{23})_{\tau 1}|0\rangle + (B_{23})_{\tau\tau}|1\rangle$$

Question: How to represent $\hat{B}_{23}|0\rangle$ a linear combination of $|0\rangle$ and $|1\rangle$?

$\hat{B}_{23}|0\rangle =$ 

$= \sum_i$  $\left[F^{\tau\tau\tau}_{\tau}\right]_{1i}$

$= \sum_i$  $\left[F^{\tau\tau\tau}_{\tau}\right]_{1i} R^{\tau\tau}_{i}$

$=$  $\underbrace{\left[F^{\tau\tau\tau}_{\tau}\right]_{1i} R^{\tau\tau}_{i} \left[F^{\tau\tau\tau}_{\tau}\right]_{ij}}_{\left[F R F^{-1}\right]_{1j}}$ $(F^{-1} = F)$

$$= (FRF^{-1})_{11} \, |0\rangle + (FRF^{-1})_{12} \, |1\rangle$$

using $F = \begin{bmatrix} 1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & -1/\phi \end{bmatrix}$ and $R^{\tau\tau}_{1} = e^{-4\pi i/5}$

$R^{\tau\tau}_{\tau} = -e^{2\pi i/5}$

One finds,

$$B_{23} = FRF^{-1}$$
$$= \begin{bmatrix} -\dfrac{e^{-i\pi/5}}{\phi} & -i\dfrac{e^{-i\pi/10}}{\sqrt{\phi}} \\ -i\dfrac{e^{-i\pi/10}}{\sqrt{\phi}} & -\dfrac{1}{\phi} \end{bmatrix}$$

Since anyons τ_1, τ_2 already have a fusion channel, the corresponding braid matrix $\hat{B}_{12}$ is trivial $\hat{B}_{12} = \begin{bmatrix} R^{\tau\tau}_{1} & \\ & R^{\tau\tau}_{\tau} \end{bmatrix}$.

One can also include the state $|2\rangle$ in the definition of the braid matrix. Since its total fusion channel is different (τ_1, τ_2, τ_3 fuse to 1 in state $|2\rangle$), it can't mix with $|0\rangle$ and $|1\rangle$. Therefore $\hat{B}_{12}$ and $\hat{B}_{23}$ only yield a phase when acting on $|2\rangle$.

One finds $\hat{B}_{12} \, |2\rangle = \hat{B}_{23} |2\rangle$

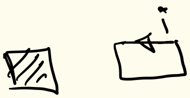
$$= R^{\tau\tau}_{\tau} \, |2\rangle = -e^{-2\pi i/5} \, |2\rangle$$

(left as homework).

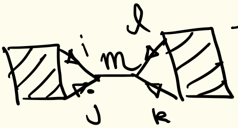
String-net phases

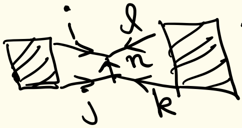
Above, we discussed space-time diagrams for anyons using R and F matrices. The idea of string-net phases is to consider a similar approach to wave-fns. Wave-fn is written as a sum of strings while allowing strings to intersect/branch.

The strings satisfy:

$\Phi\Big[$ 

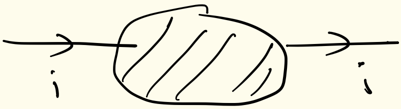
$\Big] = d_i\, \Phi\Big[$  $\Big]$

$\Phi\Big[$ 

$\Big] = \sum_n F^{ijm}_{kln}\, \Phi\Big[$ 

$\Big]$

F symbol acts in the same-way as in space-time diagrams. Other rules are more obvious, e.g., a string type doesn't change due to local processes



, similar to space-time diagrams.

F symbols again satisfy pentagon eqn.

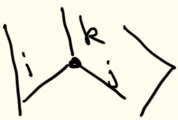
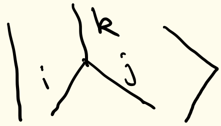
The indices i, j, k etc. take value from 0 to N where N is the number of string types.

Remarkably, more each set of valid (F, d) one can find an exactly solvable Hamiltonian of the form similar to toric code:

$$H = -\sum_I Q_I - \sum_p B_p \, .$$

The Hilbert space consists of spins that take values $0 - N$ and live on the links of honeycomb lattice. If the spin state is $i \neq 0$, one draws a string on a link with index i.

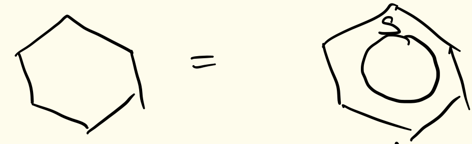
Q_I acts on a site as

Q_I  $= \delta_{ijk}$ 

where $\delta_{ijk} = 1$ if $\{i, j, k\}$ allowed string config, otherwise 0.

$$B_p = \sum_{s=0}^{N} a_s B_p^s$$ where s is the string type

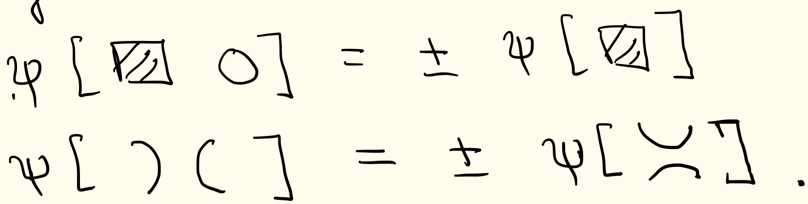
acts on a plaquette p as

B_p^s  i.e. it adds a string of type s on plaquette p. a_s are arbitrary excepts $a_{s^*} = a_s^*$ where s^* denotes a dual string with opposite orientation, a bit like anti-particle in space-time diagrams.

Example # 1:

$N=1$ (only one type of string) with no branching.

The only solns. of the pentagon eqn. are.



Thus, $\Phi(X) = (\pm)^{X_c}$ where X corresponds to a close loop config. and X_c is the number of loops in config. X, i.e.,

$$|\text{ground state}\rangle = \sum_{\mathcal{D}} (-)^{\#\text{of loops}} |\mathcal{D}\rangle$$

The + sign corresponds to $\mathbb{Z}_2$ toric code ($\mathbb{Z}_2$ guage theory) and − sign corresponds to the double semion theory whose quasiparticle content is $(1, \phi, \phi^*, m)$ where m is the bound state of ϕ and ϕ^*. The fusion rules are :

	1	ϕ	ϕ^*	m
1	1	ϕ	ϕ^*	m
ϕ	ϕ	1	m	ϕ^*
ϕ^*	ϕ^*	m	1	ϕ
m	m	ϕ^*	ϕ	1

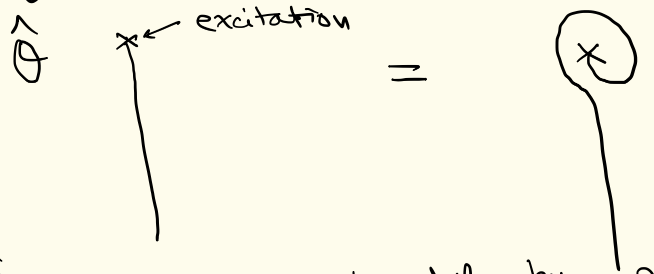
The twists (self-statistics) are $(1, i, -i, 1)$ respectively. ϕ and ϕ^* are called 'semions'.

The mutual statistics is :

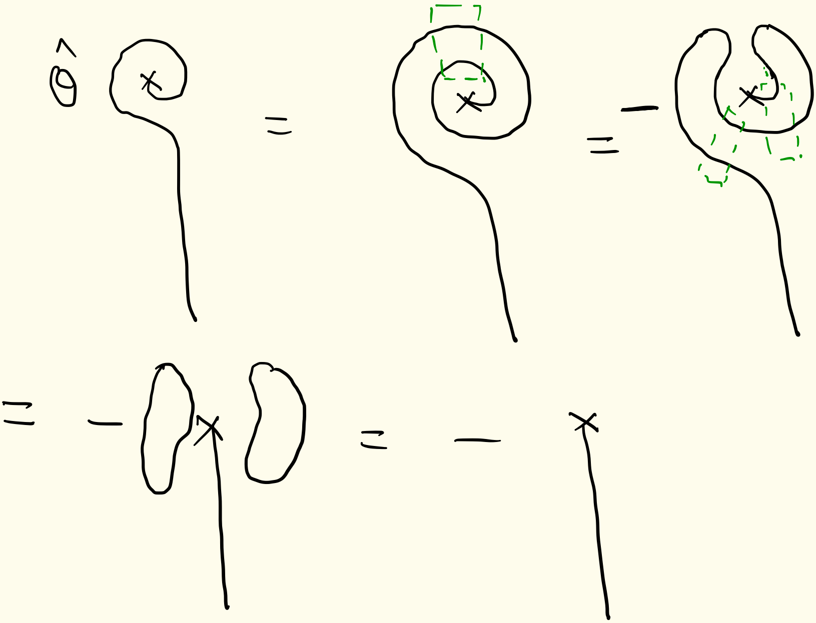
	1	ϕ	ϕ^*	m
1	1	1	1	1
ϕ	1	-1	1	-1
ϕ^*	1	1	-1	-1
m	1	-1	-1	1

Let's derive some of these results from simple considerations.

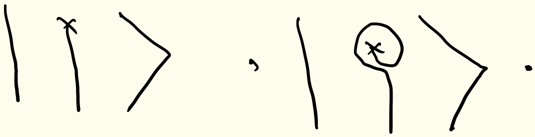
The excitations are given by ends of an open string. Define the twist operator $\hat{\Theta}$ as



That is $\hat{\Theta}$ rotates a quasiparticle by 2π, as it should. The self-statistics of quasiparticles is given by the eigenvalues of $\hat{\Theta}$.



Thus $\hat{\Theta} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = i\sigma_y$ in the basis



$\text{eig}(\hat{\Theta}) = \pm i$, which corresponds to ϕ and ϕ^*, as stated above. The bound state has exchange statistics of $i \times -i = 1$.

Given the similarity between $\mathbb{Z}_2$ toric code and double semion ground states, let's also note the Hamiltonian for the double semion:

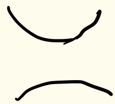
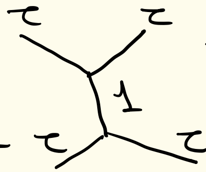
$$H = -\sum_{\text{vertices}} \prod_{Y} \sigma^x + \sum_{p} \prod_{\text{⬡}} \sigma^z \prod_{\substack{j \in \text{legs of} \\ p}} e^{\frac{i\pi}{4}\left[\frac{1-\sigma_j^x}{2}\right]}$$

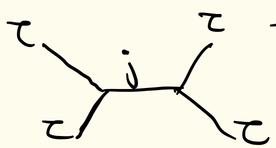
The two terms commute as one may check.

Example # 2 :

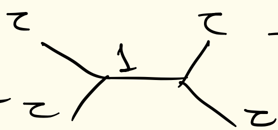
Next consider the simplest model which allows for branching, with $N = 1$. Since there is only one kind of string, one might guess this is related to the Fibonacci anyons, which is correct - it's a doubled version of that.

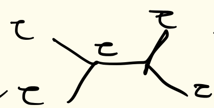
String net rules :

$\Psi\Big[$  $\Big] = \Psi\Big[$  $\Big]$

$= \sum_j \left[F^{\tau\tau\tau}_{\tau}\right]_{1j}\ \Psi\Big[$  $\Big]$

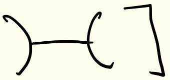
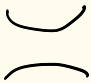
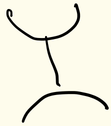
Using our earlier result

$= \left[F^{\tau\tau\tau}_{\tau}\right]_{11}\ \Psi\Big[$  $\Big]$

$+ \left[F^{\tau\tau\tau}_{\tau}\right]_{1\tau}\ \Psi\Big[$  $\Big]$

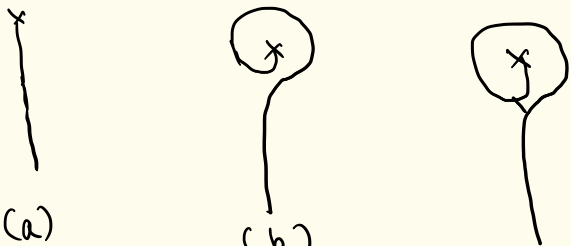
$$= \Phi^{-1}\ \Psi\left[\ )\ (\ \right] + \sqrt{\Phi^{-1}}\ \Psi\left[\ )\!-\!(\ \right]$$

Similarly,

Ψ [] = $\sqrt{\phi^{-1}}$  $-\ \phi^{-1}$ 

$[= F_{\tau 1}]$ $\qquad$ $[= F_{\tau\tau}]$

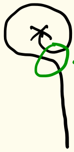
Let's use these rules to calculate self-statistics of excitations. Due to allowed branching, there are three kinds of string endings:

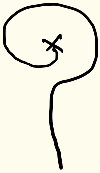
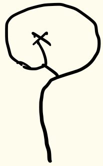


(a) $\qquad$ (b) $\qquad$ (c)

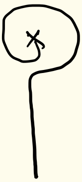
everything else can be reduced to these by F-moves.

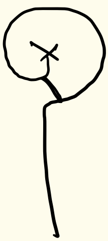
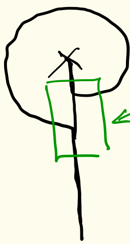
Lets apply $\hat{\Theta}$ to these three.

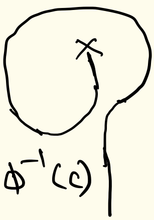
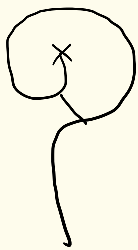
$\hat{\Theta}$  $=$  apply F here

$= \phi^{-1}$  $+ \sqrt{\phi^{-1}}$ 

$= \phi^{-1}\,(b) + \sqrt{\phi^{-1}}\,(c)$

$\hat{\Theta}$  $=$  $= (a)$

$\hat{\Theta}$  $=$  apply F here

$= \sqrt{\phi^{-1}}$  $- \phi^{-1}$ 

$= \sqrt{\phi^{-1}}\,(b) - \phi^{-1}(c)$

$$\Rightarrow \quad \hat{\theta}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & \phi^{-1} & \sqrt{\phi^{-1}} \\ 1 & 0 & 0 \\ 0 & \sqrt{\phi^{-1}} & -\phi^{-1} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Diagonalizing this, one finds

$$\mathrm{eig}(\hat{\theta}) = 1,\ e^{\pm i 4\pi/5}.$$

$e^{i4\pi/5}$ corresponds to τ and

$e^{-i4\pi/5}$ " " τ^*, it's time-reversed partner, similar to φ and φ^* in double-semion. Hence the name double-fibonacci. This is a general feature of string-net models, they are time-reversal symmetric.
